Real
Rate

Causal Analysis

With an Application to Insurance Ratings

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Part 1

Causal Analysis:

Total Derivatives and Graph Theory

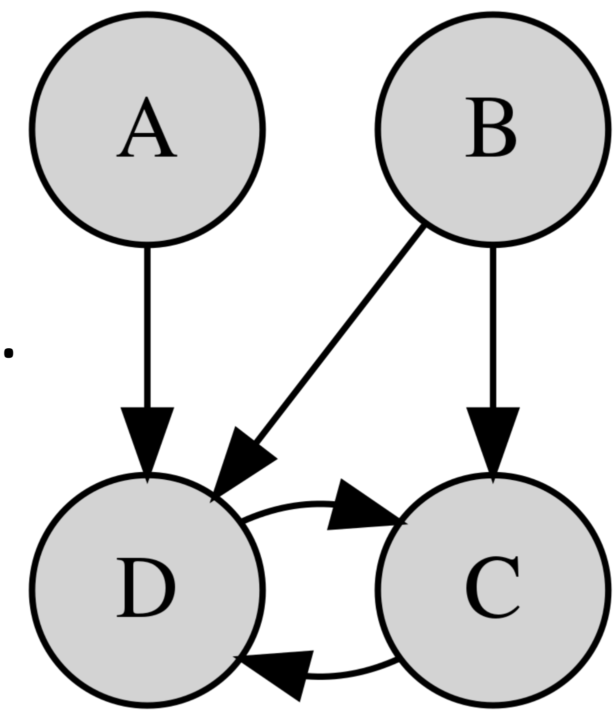
Causal Graph (1/2)

- Causal graph shows <u>causal dependencies</u> between variables (vertices / nodes).
- Arrows (edges / vertices) are causal <u>directions</u> ”→” instead of symmetric correlations.
- Causal graph of direct effects is a <u>structural</u> model, total effects are reduced form model.
- A means for explainable artificial intelligence (XAI) in neural networks

Causal Graph (2/2)

Each arrow indicates a direct cause. Example:

- D depends on A, B, C.
  C depends on B, D.
  A and B are independent.
- A, B exogenous (no arrow in).
  C, D endogenous.
- Cyclic graph since C, D
  cause each other



Causal Analysis

- Causal analysis of sensitivity:
  Effect of a variable on another
- Mediation analysis:
  Decomposition of effects over child variables

Notation

- Matrix with row and column vectors

$$\mathrm{M_y} = [\mathrm{m_{yj\iota}}]_{\mathrm{j},\iota=1,\ldots,\mathrm{n}} = \begin{bmatrix} \mathbf{m}_{\mathrm{y}(1)} \\ \vdots \\ \mathbf{m}_{\mathrm{y}(\mathrm{n})} \end{bmatrix} = [\mathbf{m}_{\mathrm{y}1}, \ldots, \mathbf{m}_{\mathrm{yn}}]$$

- $\mathrm{M_{yji}}$ is $\mathrm{M_y}$ with row j, column i replaced by zeros
- $\mathrm{I_n}$ is n-dim. identity matrix and $\mathbf{1}_\mathrm{n}$ vector of ones
- Partial derivative: $\frac{\partial \mathbf{y}}{\partial \mathbf{x}^\mathrm{T}}$
- Total derivative: $\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{x}^\mathrm{T}}$

Equation system

Model equations in matrix form:

$$\mathbf{y} = \mathrm{M}(\mathbf{y}, \mathbf{x}) = \begin{bmatrix} \mathrm{M}^1(\mathbf{y}, \mathbf{x}) \\ \vdots \\ \mathrm{M}^n(\mathbf{y}, \mathbf{x}) \end{bmatrix}$$

$$\underset{(n\times1)}{\mathbf{y}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \underset{(m\times1)}{\mathbf{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

Linearization (1/2)

Our Approach:

1) Given a system of model equations (e.g. expert system from prior knowledge)
2) We *define* effects as total derivatives
3) Using linear algebra and matrix notation to summarize all effects

Linearization (2/2)

- Total differential:

$$\mathrm{dy_j} = \sum_{\mathrm{h}=1}^{\mathrm{n}} \frac{\partial \mathrm{M^j}}{\partial \mathrm{y_h}} \mathrm{dy_h} + \sum_{\mathrm{l}=1}^{\mathrm{m}} \frac{\partial \mathrm{M^j}}{\partial \mathrm{x_l}} \mathrm{dx_l}\,, \mathrm{j} = 1, \dots, \mathrm{n}$$

- In matrix form:

$\mathrm{d}\mathbf{y} = \mathrm{M_y}\mathrm{d}\mathbf{y} + \mathrm{M_x}\mathrm{d}\mathbf{x}$ (structural form)

$\Rightarrow \mathrm{d}\mathbf{y} = \left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1}\mathrm{M_x}\,\mathrm{d}\mathbf{x}$ (reduced form)

- Normalization ($\mathrm{y_j}$ to left hand side):

$\mathrm{M_y}$ has zero main diagonal

Total (Exogenous) Derivative

Dividing total differential by $\mathrm{dx_i}$:

$$\frac{\mathrm{d}y_j}{\mathrm{d}x_i} = \sum_{h=1}^{n} \frac{\partial M^j}{\partial y_h}\frac{\mathrm{d}y_h}{\mathrm{d}x_i} + \sum_{l=1}^{m} \frac{\partial M^j}{\partial x_l}\frac{\mathrm{d}x_l}{\mathrm{d}x_i}$$

$$= \sum_{h=1}^{n} \frac{\partial M^j}{\partial y_h}\frac{\mathrm{d}y_h}{\mathrm{d}x_i} + \frac{\partial M^j}{\partial x_i}$$

In matrix form:

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{x}^T} = M_y \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{x}^T} + M_x$$

$$\Rightarrow \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{x}^T} \equiv \boxed{E_x = \left(I_n - M_y\right)^{-1} M_x}$$

Total Endogenous Derivative (1/2)

With own effects defined as one ($\mathrm{dy}_\iota/\mathrm{dy}_\iota \equiv 1$) and Kronecker delta $\delta_{j\iota}$:

$$\frac{\mathrm{dy}_j}{\mathrm{dy}_\iota} = \left(1 - \delta_{j\iota}\right) \sum_{h=1}^{n} \frac{\partial M^j}{\partial y_h} \frac{\mathrm{dy}_h}{\mathrm{dy}_\iota} + 1 \cdot \delta_{j\iota}$$

In matrix form, with element wise product ($\circ$):

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{y}^T} &= (1_{nn} - I_n) \circ \left(M_y \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{y}^T} \right) + I_n \\
&= M_y \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{y}^T} - I_n \circ \left(M_y \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{y}^T} \right) + I_n
\end{aligned}$$

Total Endogenous Derivative (2/2)

$$\Leftrightarrow \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{y}^{\mathrm{T}}} = \left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1} \left(\mathrm{I_n} \circ \left(\mathrm{M_y} \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{y}^{\mathrm{T}}}\right) + \mathrm{I_n}\right)$$

Solving for $\mathrm{d}\mathbf{y}/\mathrm{d}\mathbf{y}^{\mathrm{T}}$ whilst ensuring unit main diagonal (own effects).

Solution is given by column-wise normalization:

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{y}^{\mathrm{T}}} \equiv \boxed{\mathrm{E_y} = \left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1} \left(\mathrm{I_n} \circ \left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1}\right)^{-1}}$$

Final Effects

- For mediation analysis we decompose an effect over its outgoing edges to its child variables.
- Each effect is the sum of its final effects e_{jhi} from node x_j over y_h to the j-th „final“ variable of interest y_j. (e.g. economic capital.)
- Note, we need the structural parameters / direct effects M_x and M_y.

Final Exogenous Effects

The (j,i)-th effect of $E_x = (I_n - M_y)^{-1} M_x$,

$$\frac{dy_j}{dx_i} = \left((I_n - M_y)^{-1}\right)_{(j)} \mathbf{m}_{xi} = \sum_{h=1}^{n} e_{jhi,x}$$

can be decomposed into the sum of its final effects $e_{jhi,x}$ from x_i over y_h to final variable y_j:

$$e_{jhi,x} = \left((I_n - M_y)^{-1}\right)_{jh} \frac{\partial M^h}{\partial x_i}$$

In matrix form:

$$\boxed{F_x^j = \left(\left(\left((I_n - M_y)^{-1}\right)_{(j)}\right)^T \mathbf{1}_{(m)}\right) \circ M_x}$$

Final Endogenous Effects

The j-th effect of $\mathbf{e}_{y\iota\iota} = \left(\mathrm{I_n} - \mathrm{M}_{y\iota\iota}\right)^{-1}\mathbf{m}_{y\iota\iota}$ with $\mathrm{e}_{y\iota\iota} \equiv 1$,

$$\frac{\mathrm{d}y_j}{\mathrm{d}y_\iota} = \left(1 - \delta_{j\iota}\right)\left(\left(\mathrm{I_n} - \mathrm{M}_{y\iota\iota}\right)^{-1}\right)_{(j)} \mathbf{m}_{y\iota\iota} + \delta_{j\iota} = \sum_{h=1}^{n} \mathrm{e}_{jh\iota,y}$$

(see Bartel (2019), sec. 3.4) can be decomposed into sum of its final effects $\mathrm{e}_{jh\iota,y}$ from y_ι over y_h to final variable y_j:

$$\mathrm{e}_{jh\iota,y} = \left(\left(\mathrm{I_n} - \mathrm{M}_{y\iota\iota}\right)^{-1}\right)_{jh} \frac{\partial \mathrm{M}^h}{\partial y_\iota} \text{ with } \mathrm{e}_{\iota\iota\iota,y} \equiv 0$$

In matrix form:

$$\boxed{\mathrm{F}_y^j = \left(1_{nn} - \mathrm{I_n}\right) \circ \left[\left(\left(\mathrm{I_n} - \mathrm{M}_{y11}\right)^{-1}\right)_{(j)}^{\mathrm{T}}, \ldots, \left(\left(\mathrm{I_n} - \mathrm{M}_{ynn}\right)^{-1}\right)_{(j)}^{\mathrm{T}}\right] \circ \mathrm{M}_y}$$

Summary: Effect Formulas

Total exogenous and endogenous effects:

$$\mathrm{E_x} = \left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1}\mathrm{M_x}$$

$$\mathrm{E_y} = \left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1}\left(\mathrm{I_n} \circ \left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1}\right)^{-1}$$

Final exogenous and endogenous effects on $\mathrm{y_j}$:

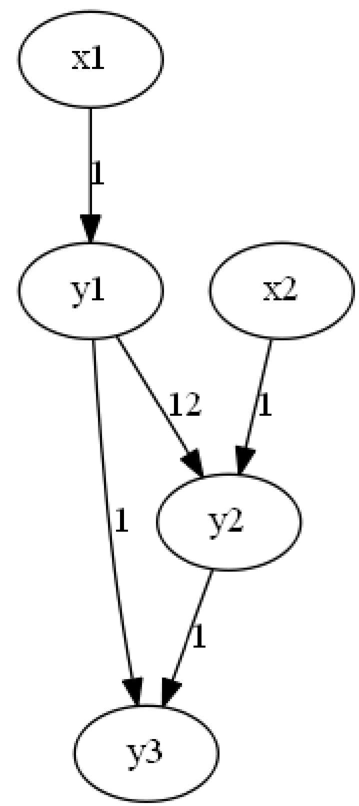
$$\mathrm{F_x^j} = \left(\left(\left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1}\right)_{(\mathrm{j})}^{\mathrm{T}} \mathbf{1}_{(\mathrm{m})}\right) \circ \mathrm{M_x}$$

$$\mathrm{F_y^j} = \left(1_{\mathrm{nn}} - \mathrm{I_n}\right) \circ \left[\left(\left(\mathrm{I_n} - \mathrm{M_{y11}}\right)^{-1}\right)_{(\mathrm{j})}^{\mathrm{T}}, \dots, \left(\left(\mathrm{I_n} - \mathrm{M_{ynn}}\right)^{-1}\right)_{(\mathrm{j})}^{\mathrm{T}}\right] \circ \mathrm{M_y}$$

Example (1/7)

- $y_1 = x_1$
  $y_2 = 2y_1^2 + x_2$
  $y_3 = y_1 + y_2$
  Note: read (=) as (←)
- Exogenous $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
- Solution $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ 23 \end{bmatrix}$
- Partial graph: denote edges by non-zero partial derivatives M_x, M_y at $\mathbf{x}$

<u>Partial Graph</u>



Example (2/7)

- Simple nonlinear, acyclic system with non-constant matrix of partial derivatives:

$$\underset{(\mathrm{n}\times\mathrm{m})}{\mathrm{M_x}} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}^{\mathrm{T}}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

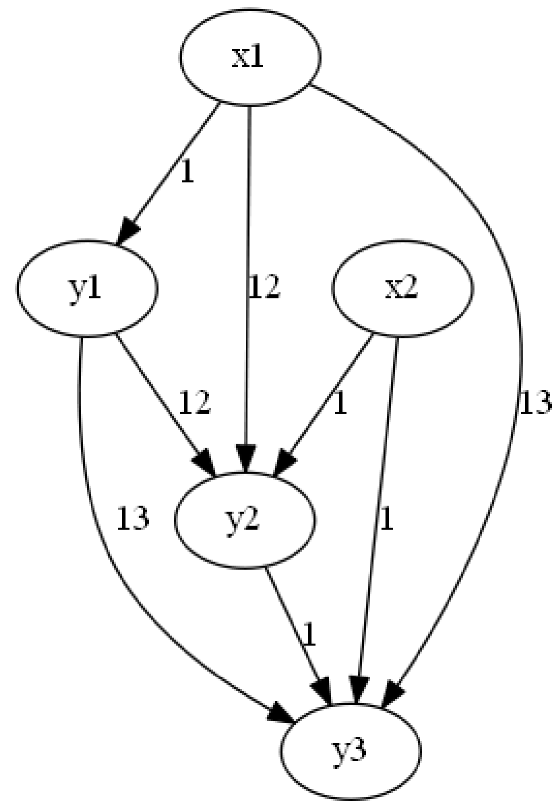
$$\underset{(\mathrm{n}\times\mathrm{n})}{\mathrm{M_y}} = \frac{\partial \mathbf{y}}{\partial \mathbf{y}^{\mathrm{T}}} = \begin{bmatrix} 0 & 0 & 0 \\ 4\mathrm{x}_1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 12 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

- Proxy: original model is neither linear nor homogeneous (dependence on individual $\mathbf{x}$)

Example (3/7)

- Total graph: denote edges by non-zero total effects $\mathrm{E_x}$, $\mathrm{E_y}$ at $\mathbf{x}$
- Don't show total own effects ($\equiv 1$)
- More arrows than partial graph because of indirect effects

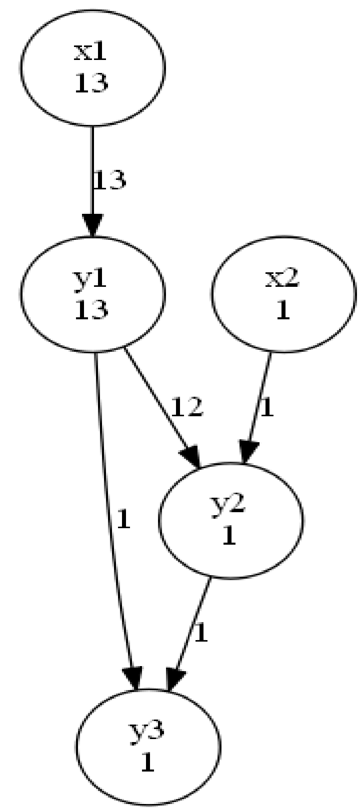
Total Graph



Example (4/7)

- Mediation analysis for final variable $\mathrm{y_j}$ with $\mathrm{j} = 3$
- Final effect: is total effect, partitioned over all outgoing edges
- Final graph: denote edges by non-zero final effects $\mathrm{F_x^j}$, $\mathrm{F_y^j}$ at $\mathbf{x}$, own effects are defined to be zero, denote nodes by total effects

Final Graph



Example (5/7)

Total Effects:

$$\mathrm{E_x} = \left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1}\mathrm{M_x} = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 13 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 4\mathrm{y_1} & 1 \\ 4\mathrm{y_1} + 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 12 & 1 \\ 13 & 1 \end{bmatrix}$$

$$\mathrm{E_y} = \left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1}\left(\mathrm{I_n} \circ \left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1}\right)^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4\mathrm{y_1} & 1 & 0 \\ 4\mathrm{y_1} + 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 13 & 1 & 1 \end{bmatrix}$$

Example (6/7)

Final Effects:

$$\mathrm{F_x^3} = \left(\left(\left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1}\right)^{\mathrm{T}}_{(3)} \mathbf{1}_{(\mathrm{m})}\right) \circ \mathrm{M_x}$$

$$= \begin{bmatrix} 13 & 13 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

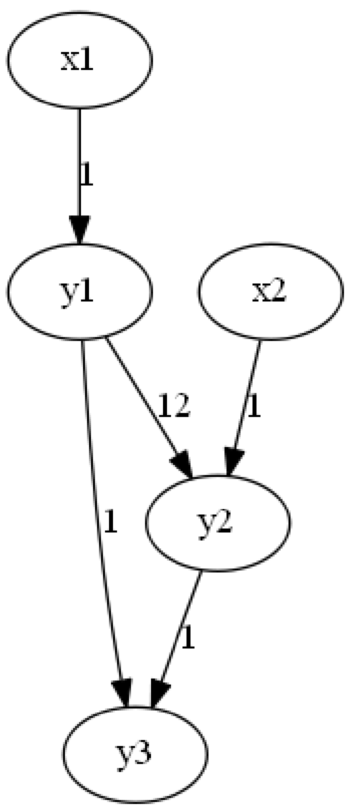
$$\mathrm{F_y^3} = (1_{\mathrm{nn}} - \mathrm{I_n}) \circ \left[\left(\left(\mathrm{I_n} - \mathrm{M_{y11}}\right)^{-1}\right)^{\mathrm{T}}_{(3)}, \dots, \left(\left(\mathrm{I_n} - \mathrm{M_{ynn}}\right)^{-1}\right)^{\mathrm{T}}_{(3)}\right] \circ \mathrm{M_y}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 12 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Example (7/7)

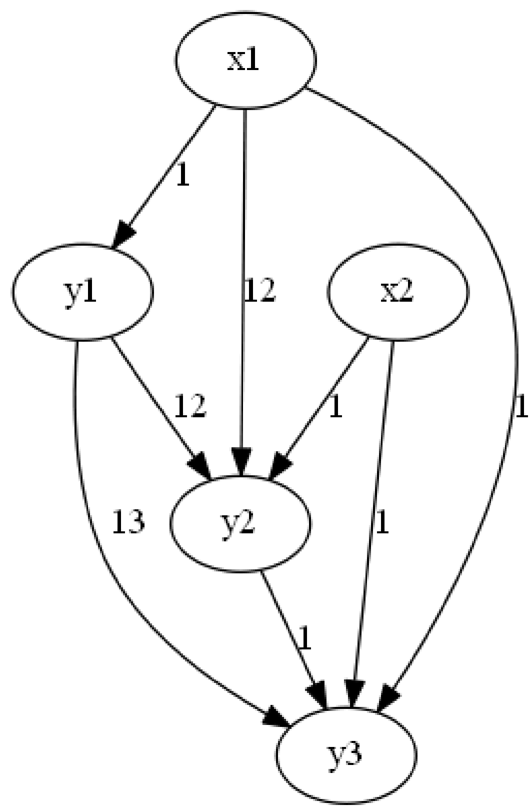
Partial Graph

(direct effects)



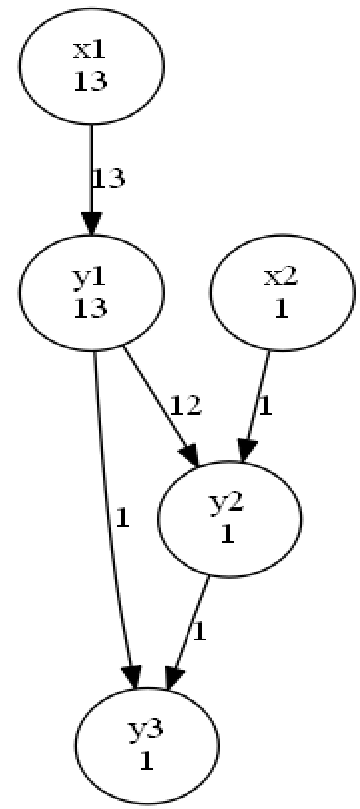
Total Graph

(total effects)



Final Graph

(mediation effects on final variable y_3)



Special Case: DAG (1/3)

In a directed acyclic graph (DAG) we have:

- Total effects are sum of products of direct effects over all paths. In our example for the total effect of x_1 on y_3 we have paths $(\mathrm{x}_1 \rightarrow \mathrm{y}_1 \rightarrow \mathrm{y}_3)$ and $(\mathrm{x}_1 \rightarrow \mathrm{y}_1 \rightarrow \mathrm{y}_2 \rightarrow \mathrm{y}_3)$ giving $(1 * 1) + (1 * 12 * 1) = 13.$
- M_y is strictly lower triangular after topological sorting. Thus $\left(\mathrm{I}_\mathrm{n} - \mathrm{M}_\mathrm{y}\right)$ is unitriangular and also its inverse. Thus normalizing factor $\left(\mathrm{I}_\mathrm{n} \circ \left(\mathrm{I}_\mathrm{n} - \mathrm{M}_\mathrm{y}\right)^{-1}\right)^{-1} = \mathrm{I}_\mathrm{n}.$

Special Case: DAG (2/3)

- E_y is finite Neumann series:

$$E_y = \left(I_n - M_y\right)^{-1} = I_n + \sum_{k=1}^{n} M_y^k$$

In our example we have

$$E_y = I_3 + M_y + M_y^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 12 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 12 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4y_1 & 1 & 0 \\ 4y_1 + 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 13 & 1 & 1 \end{bmatrix}$$

Special Case: DAG (3/3)

- Final endogenous effect:

$$F_y^j = \left(\left(\left(I_n - M_y \right)^{-1} \right)_{(j)}^T \mathbf{1}_{(n)} \right) \circ M_y$$

In our example we have:

$$F_y^3 = \begin{bmatrix} 13 & 13 & 13 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 0 \\ 12 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 12 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Graph Theoretic Effects (1/3)

However, in graph theory, effects are defined using Pearl‘s <u>counterfactual</u> analysis: To answer

“Y would be y had Z been z“

replace equation for variable Z by constant z. This amounts to deleting all edges going into node Z: In this modified model $\mathrm{M}_{\mathrm{x}(\iota)}$, $\mathrm{M}_{\mathrm{y}(\iota)}$ row ι is replaced by zeros. Since Z now is exogenous, we can compute the derivative w.r.t. Z.

See Definition 4 in Pearl (2009).

Graph Theoretic Effects (2/3)

Definition:

The graph theoretic effect is the <u>derivative</u> of Y w.r.t ι-th variable Z in the <u>modified model</u> $\mathrm{M}_{\mathrm{x}(\iota)}$, $\mathrm{M}_{\mathrm{y}(\iota)}$ with edges going into Z being set to zero and Z set to constant z.

- Question: Is the graph theoretic effect *identical* to the total derivative?
- Answer: YES! See Bartel (2019), sec. 3.2.

Graph Theoretic Effects (3/3)

1) Graph theoretic effect is not conditional expectation $\mathrm{E(Y|z)}$ but $\mathrm{E\big(Y|do(z)\big)}$ in the counterfactual model using the do-Operator.
2) Having $\iota = 1, \dots, \mathrm{n}$ *different* modified models.
3) The nulled out ι-th model equation is of no importance for the effects w.r.t. that variable.
4) Bollen (1989), formula (8.81) *defines* $\mathrm{E_y}$ as $\left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1} - \mathrm{I_n}$. Own effects $\neq 1$ if cyclic.

Graph Theoretic Interpretation

- Inverse can be written as Neumann series:

$$\left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1} = \mathrm{I_n} + \sum_{\mathrm{k=1}}^{\infty} \mathrm{M_y^k}$$

- Adjacency matrix:

$$\mathrm{A} = \mathrm{M_y^T}$$

- Binary versions of $\mathrm{M_x}$, $\mathrm{M_y}$ give Identification matrices $\mathrm{ID_x}$, $\mathrm{ID_y}$ (1 for non-zero element, 0 otherwise).

Part 2

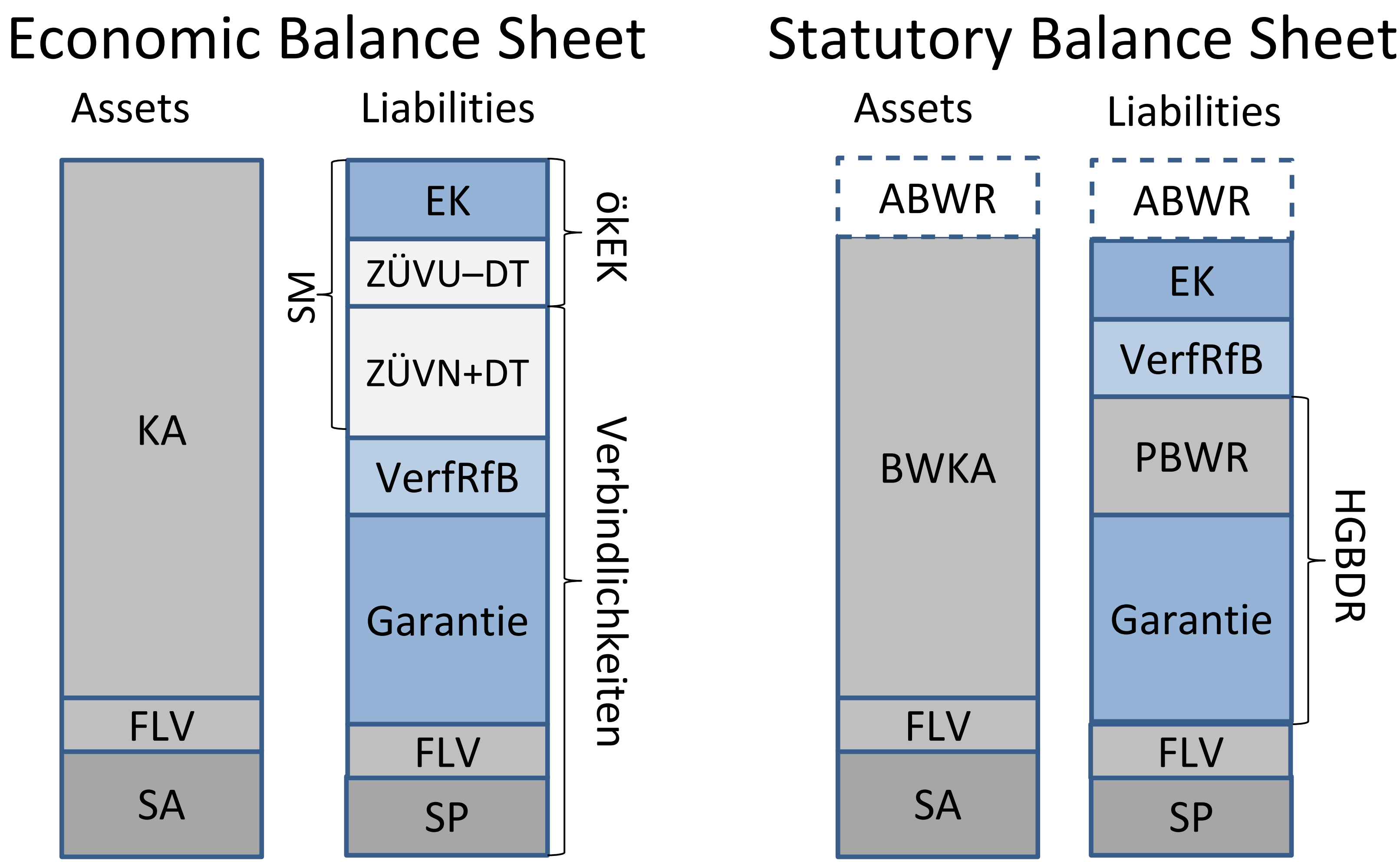
Application:

Insurance Financial Strength Ratings

Application: Insurance Ratings

- Application to financial strength ratings of German life insurers
- Model equations from expert system
- Holistic approach instead of just key figures
- Public input data from balance sheet
- Revaluation to market values
- Simple: no simulation, no cash flows
- Final variable is economic capital ratio (ökonomische Eigenkapital-Quote, ökEK)

Economic Balance Sheet



Model Equations (1/2)

1) EK = EKoGRNV + GR + NV
2) VerfRfB = FreieRfB + SÜAF
3) DR = HGBDR – ZZR
4) MRZ = (ZA – ZZRA) / DR
5) KE = KAE – KAA
6) ZE = KE – ZA
7) JÜV = JÜ + GewAb + Steuer
8) RÜ = JÜV + ZRfB + DG
9) RÜE = RÜ – ZE
10) BABRate = ZVF / (DR + FLV)
11) D = 1 / (BABRate + R)
12) KA = BWKA + ABWR
13) Assets = BS + ABWR
14) MWDR = DR * (1 + D*(MRZ–R))
15) ZÜVT = RÜE * D
16) ZÜKA = DR – MWDR + ZZR
17) PBWR = ZÜKA + ZÜVT
18) Garantie = HGBDR – PBWR
19) ZÜ = ABWR + PBWR
20) GuO = … *see Bartel (2014)*
21) ZÜVU = ZÜVUdet – GuO
22) ZÜVN = ZÜ – ZÜVU
23) DT = TaxRate * ZÜVU
24) Puffer = ZÜVN + VerfRfB + DT

Model Equations (2/2)

25) ökEK = EK + ZÜVU – DT

26) ökEK-Quote = ökEK / BS

27) SM = EK + VerfRfB + ZÜ

28) SM-Quote = SM / BS

29) NVZ = KE / BWKA

30) GVZ = MRZ + ZÜVN / (DR * D)

31) SA = BS – BWKA – FLV

32) SP = BS – EK – VerfRfB – HGBDR – FLV

Note to equation 20: On how to compute the value of guarantees and options GuO = f(KA, ZÜ, VerfRfB, D; ABWR, R) for German life insurers, see Bartel (2014).

Variables (1/4)

Input

- ABWR: aktivische Bewertungsreserven / hidden reserves
- BS: HGB-Bilanzsumme / statutory total assets
- BWKA: Buchwert Kapitalanlagen / statutory investments
- DG: Direktgutschrift / direct credit
- EKoGRNV: HGB-Eigenkapital ohne GR und NV / pure equity
- FLV: Fondsgebundene LV / unit-linked insurance funds
- FreieRfB: freie RSt. für Beitragsrückerstattung / free surplus fund
- GewAb: Gewinnabführung / dividend payments
- GR: Genußrechte / participation rights
- HGBDR: HGB-Deckungsrückstellung / statutory technical reserves
- JÜ: JÜ nach Steuern, Gewinnabführung / net annual surplus
- KAA: Kapitalanlage-Aufwendungen / investment expenses
- KAE: Kapitalanlage-Erträge / investment returns

Variables (2/4)

- NV: nachrangige Verbindlichkeiten / subordinated liabilities
- R: risikoloser Marktzins / risk-free interest rate
- Steuer: Steuern / taxes
- SÜAF: Schlussüberschussanteil-Fonds / terminal bonsu fund
- ZA: Zinsaufwand / interest expenses
- ZRfB: Zuführung zur RfB / allocation to surplus fund
- ZVF: Zahlungen Versicherungsfälle / insurance benefits
- ZZR: Zinsszusatzreserve / additional interest reserve
- ZZRA: ZZR-Aufwand / expenses for additional interest reserve

<u>Output</u>

- Assets: Marktwert-Bilanzsumme / market value of assets
- BABRate: Bestandsabbaurate / lapse and termination rate
- DR: HGBDR ohne ZZR / statutory techn. reserves w/o ZZR
- DT: latente Steuern / latent taxes

Variables (3/4)

- EK: HGB-Eigenkapital / equity
- GuO: Garantien und Optionen / guarantees and options
- GVZ: nachhaltige Gesamtverzinsung / customer’s total yield
- JÜV: JÜ vor Steuern und Gewinnabführung / annual surplus
- KA: Marktwert Kapitalanlagen / market value investments
- KE: Kapitalergebnis / investment earnings
- MRZ: mittlerer Tarifrechnungszins / avg. guaranteed rate
- MWDR: Marktwert-DR / market value of technical provisions
- NVZ: Nettoverzinsung / net return
- ökEK: ökonomisches Eigenkapital / economic capital
- ÖkEK-Quote: ökonomische Eigenkapitalquote / economic capital ratio
- PBWR: passivische BWR/ hidden reserves in liabilities
- D: Passivduration / duration of liabilities
- Puffer: Puffer aus angesammelten und zuk. Gewinnen / buffers

Variables (4/4)

- RÜ: Rohüberschuss / gross surplus
- RÜE: Risiko- und Übriges Ergebnis / risk and other result
- SA: sonstige Aktiva / other assets
- SP: sonstige Passiva / other liabilities
- SM: Sicherheitsmittel / solvency margin
- SM-Quote: Sicherheitsmittelquote / statutory solvency ratio
- VerfRfB: verfügbare RfB / available surplus fund
- ZE: Zinsergebnis / interest earnings
- ZÜ: zukünftige Überschüsse / future surplus
- ZÜKA: zukünftige pass. Zinsüberschüsse / future interest surplus
- ZÜVT: zukünftige pass. vt. Überschüsse / future technical surplus
- ZÜVN: zukünftige Überschussbet. / future discretionary benefits
- ZÜVU: zukünftige Aktionärsgewinne / future dividends

Example: Insurance Economic Capital

- Financial strength rating
- HUK-COBURG Lebensversicherung AG
- Accounting year 2017
- Final effects with respect to market mean

Economic and Statutory Balance Sheet

Statutory balance sheet

Aktiva	in Mio. Euro	Passiva	in Mio. Euro
Kapitalanlagen	8.903	HGB-Eigenkapital	601
Fondsgebundene LV	158	verfügbare RfB	55
Sonst. Aktikva	375	Deckungsrückstellung	7.381
		Fondsgebundene LV	158
		Sonst. Passiva	1.240
Aktiva	9.436	Passiva	9.436

Economic balance sheet

Aktiva	in Mio. Euro	Passiva	in Mio. Euro
Kapitalanlagen	9.664	ök. Eigenkapital	938
Fondsgebundene LV	158	Puffer	1.578
Sonst. Aktikva	375	Garantie	6.283
		Fondsgebundene LV	158
		Sonst. Passiva	1.240
Aktiva	10.197	Passiva	10.197

Economic Capital

Economic Capital

Komponente ök. EK	in Mio. Euro	in % der HGB-Bilanzsumme
HGB-Eigenkapital	600,83	6,37
zukünftige Aktionärsgewinne	450,19	4,77
Steuern	-112,55	-1,19
ökonomisches Eigenkapital	938,47	9,95

Buffer

Puffer-Komponente	in Mio. Euro	in % der HGB-Bilanzsumme
verfügbare RfB	55	0,59
Überschussbeteiligung	1.410	14,94
Steuern	113	1,19
Puffer	1.578	16,72

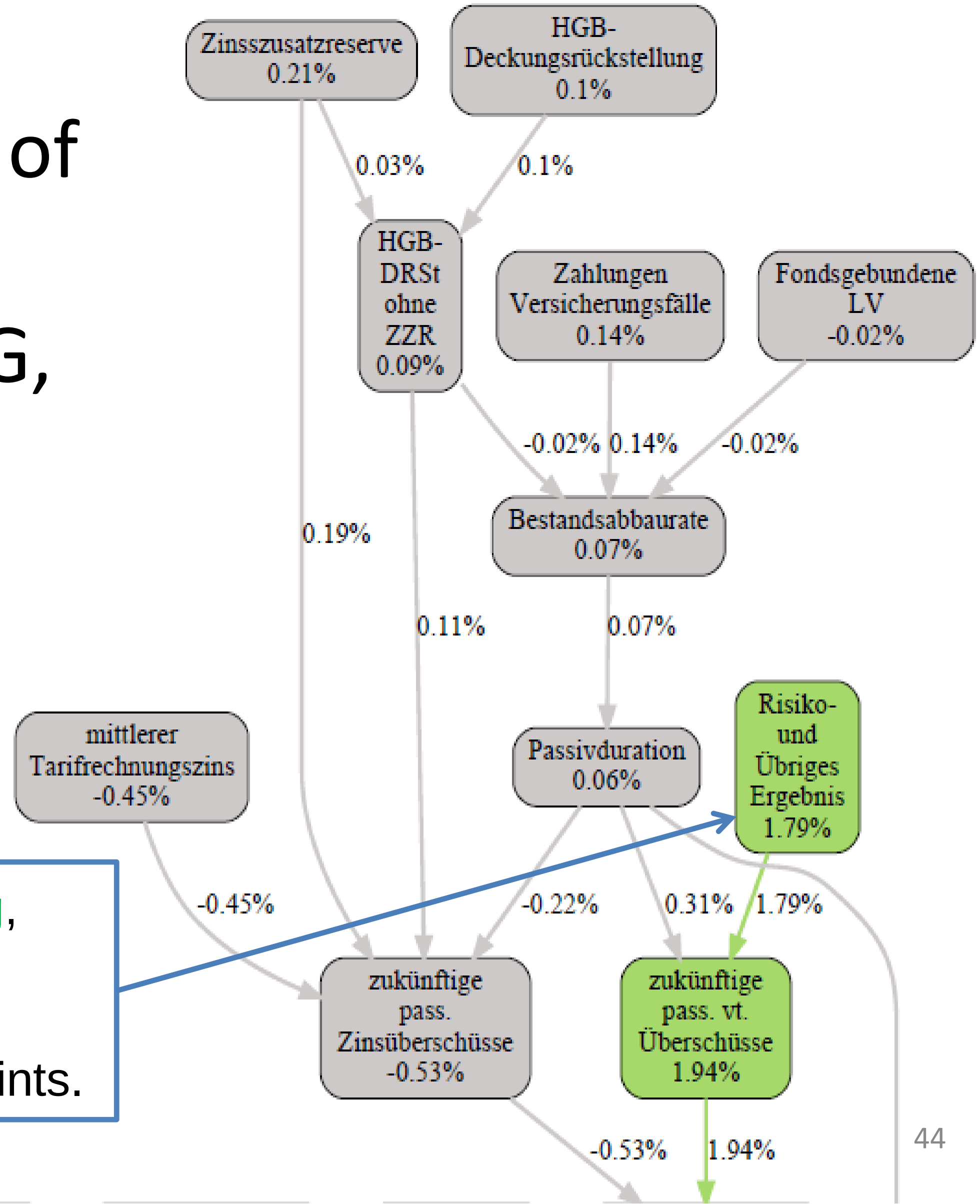
Solvency Margin

(= economic capital + buffer)

Komponente Sicherheitsmittel	in Mio. Euro	in % der HGB-Bilanzsumme
HGB-Eigenkapital	601	6,37
verfügbare RfB	55	0,59
zukünftige Überschüsse	1.860	19,71
Sicherheitsmittel	2.516	26,66

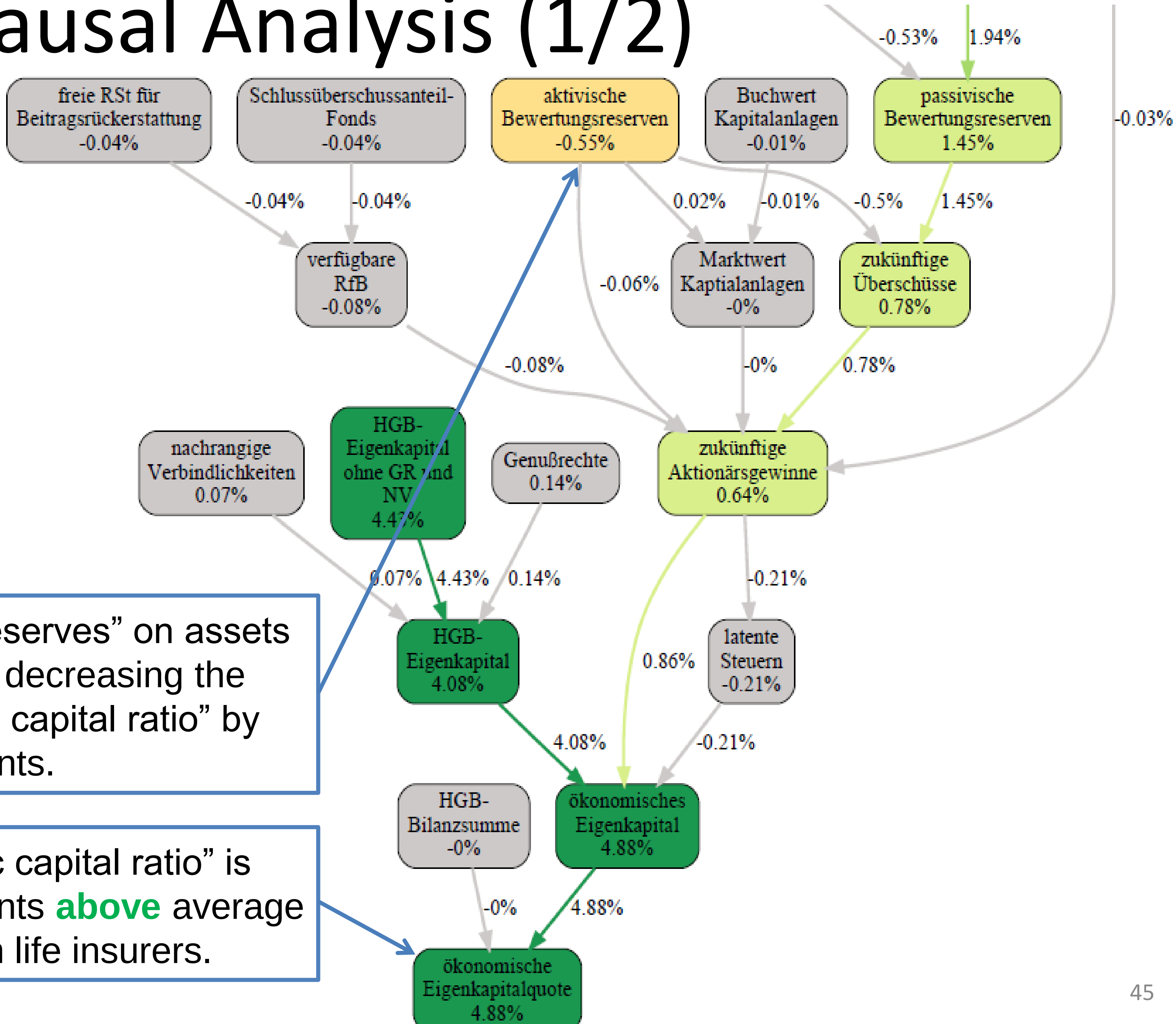
Graphical Causal Analysis (1/2)

Final graph for the economic capital ratio of HUK-COBURG Lebensversicherung AG, Accounting year 2017



"Risk and other result" is very **strong**, compared to average of German life insurers, increasing the final variable "economic capital ratio" by 1.79%-points.

Causal Analysis (1/2)



"Hidden reserves" on assets are **weak**, decreasing the "economic capital ratio" by 0.55%-points.

"Economic capital ratio" is 4.88%-points **above** average of German life insurers.

Final Effects: Strengths

Größe[0]	Rang[1]	HUK	Markt[2]	Effekt[3] ökonomische Eigenkapitalquote
Quote HGB-Eigenkapital ohne GR und.	1	6,16%	1,73%	4,43%
Quote HGB-Eigenkapital	1	6,37%	2,29%	4,08%
Quote zukünftige pass. vt. Übersch.	4	17,34%	7,87%	1,94%
Quote Risiko- und Übriges Ergebnis	5	1,37%	0,68%	1,79%
Quote passivische Bewertungsreserv.	6	11,64%	4,55%	1,45%
Quote zukünftige Überschüsse	16	19,71%	15,88%	0,78%
Quote zukünftige Aktionärsgewinne	17	4,77%	3,91%	0,64%
Quote Zinszusatzreserve	14	6,63%	5,71%	0,21%
Quote Zahlungen Versicherungsfälle	51	5,33%	6,59%	0,14%
Quote Genußrechte	4	0,14%	0,00%	0,14%
...	...	...	...	...

[0] Ratio with respect to total assets
[1] out of 59 insurers, descending order
[2] Median
[3] Change in percentage points

Final Effects: Weaknesses

Größe[0]	Rang[1]	HUK	Markt[2]	Effekt[3] ökonomische Eigenkapitalquote
...	...	...	...	...
Quote sonstige Passiva	3	13,14%	5,04%	0,00%
Quote Buchwert Kapitalanlagen	22	94,35%	92,17%	-0,01%
Quote Fondsgebundene LV	45	1,67%	4,59%	-0,02%
Quote Schlussüberschussanteil-Fond.	56	0,18%	1,54%	-0,04%
Quote freie RSt für Beitragsrücker.	57	0,41%	2,12%	-0,04%
Quote verfügbare RfB	59	0,59%	3,73%	-0,08%
Quote latente Steuern	17	1,19%	0,98%	-0,21%
mittlerer Tarifrechnungszins	14	3,27%	3,02%	-0,45%
Quote zukünftige pass. Zinsübersch.	52	-5,69%	-3,09%	-0,53%
Quote aktivische Bewertungsreserve.	47	8,07%	10,51%	-0,55%

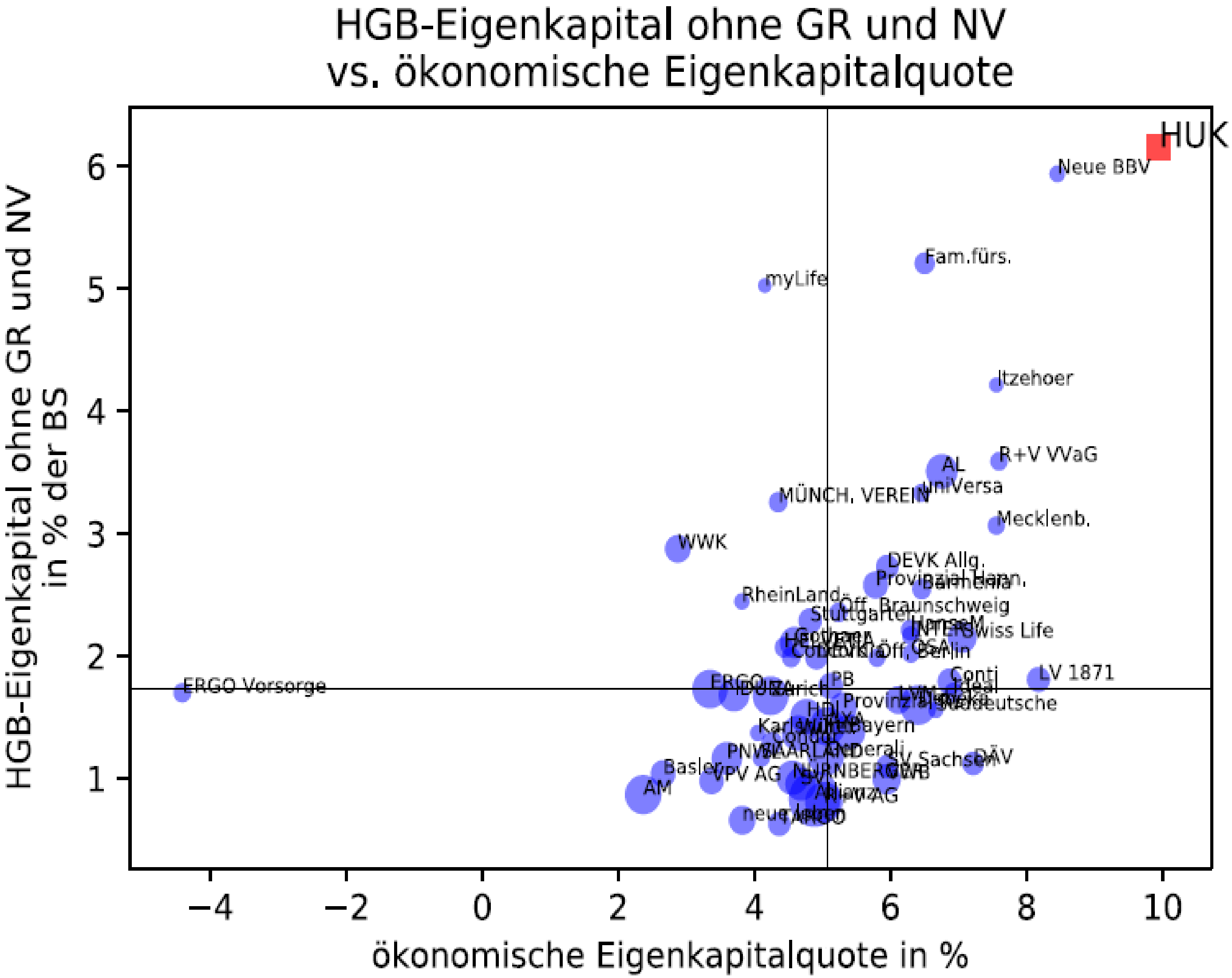
[0] Ratio with respect to total assets

[1] out of 59 insurers, descending order

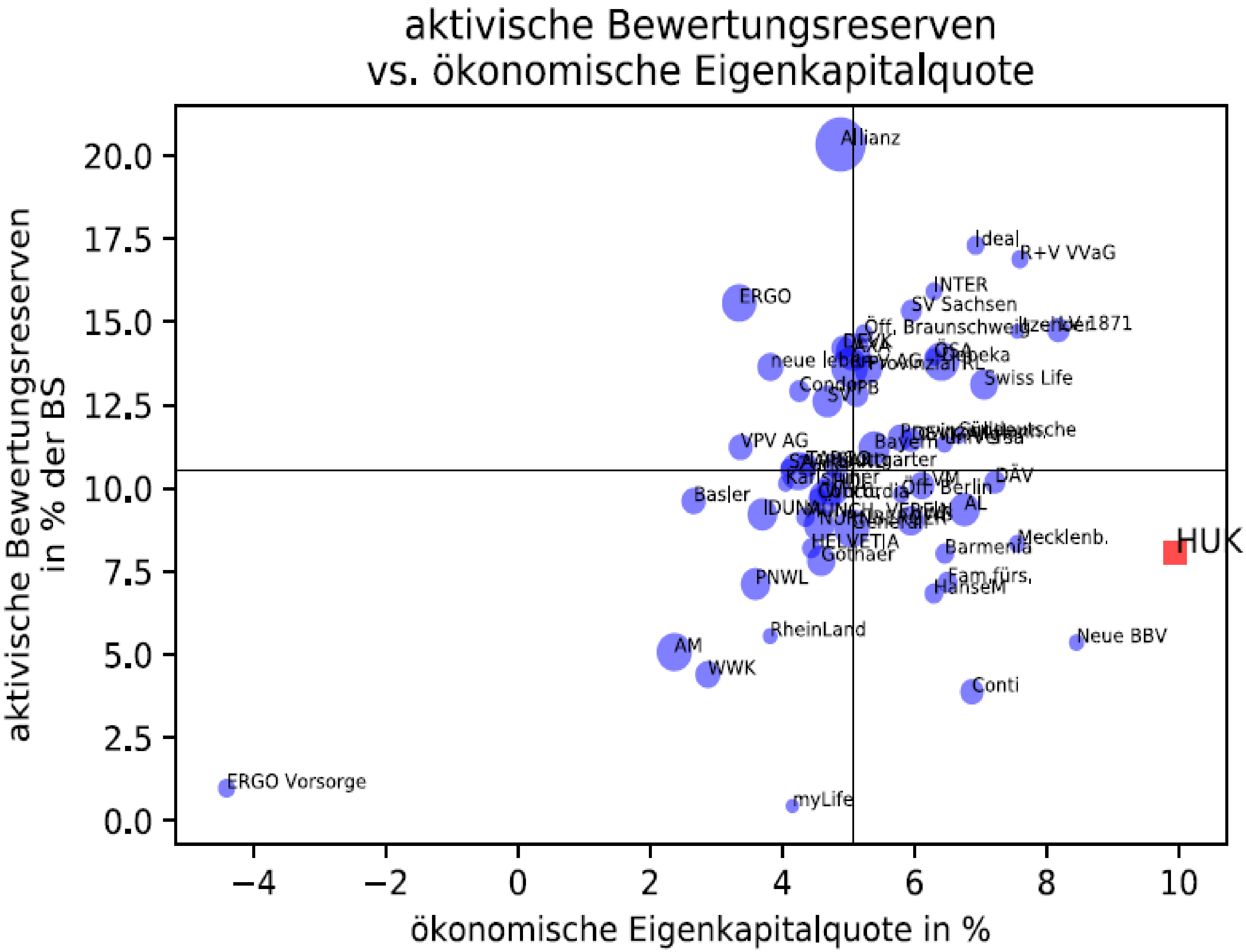
[2] Median

[3] Change in percentage points

Market Plot: Top Strength



Market Plot: Top Weakness



Input Data and Market

Größe	Rang[1]	HUK	Min.	5%[2]	50%[2]	95%[2]	Max.
aktivische Bewertungsreserven	29	761	2	52	694	6.698	43.177
Buchwert Kapitalanlagen	23	8.903	159	653	5.493	42.295	201.429
Direktgutschrift	11	48,73	0,00	0,00	1,30	141,38	292,78
Fondsgebundene LV	36	158	0	1	570	7.451	17.047
freie RSt für Beitragsrückerstattu.	44	38	2	7	152	975	7.494
Genußrechte	3	13,00	0,00	0,00	0,00	4,06	150,00
Gewinnabführung	24	0,00	0,00	0,00	0,00	125,88	381,00
HGB-Bilanzsumme	24	9.436	453	817	6.514	46.859	212.307
HGB-Deckungsrückstellung	24	7.381	141	590	4.671	38.020	186.414
HGB-Eigenkapital ohne GR und NV	7	581	11	20	116	752	1.764
Jahresüberschuss nach Steuern und.	4	34,30	-0,00	-0,00	1,84	35,07	166,00
Jahresüberschuss vor Steuern und G.	13	32,61	-15,07	-1,30	8,11	182,54	595,87
Kapitalanlage-Aufwendungen	32	15	0	1	20	144	1.710
Kapitalanlage-Erträge	24	382	9	24	283	2.019	10.943
mittlerer Tarifrechnungszins	14	3,27%	1,68%	2,23%	3,02%	3,44%	3,60%
nachrangige Verbindlichkeiten	23	7,00	0,00	0,00	0,00	314,76	888,17
Risiko- und Übriges Ergebnis	14	129,51	-10,31	3,00	38,27	378,99	517,07
Rohüberschuss	16	141	-2	3	58	505	2.649
Schlussüberschussanteil-Fonds	51	17	1	4	100	785	3.358
Steuern	48	-1,69	-63,26	-17,40	1,13	47,04	60,42
Zahlungen Versicherungsfälle	28	503	19	56	423	3.404	11.830
Zinsaufwand	23	355	4	23	223	1.796	7.101
Zinszusatzreserve	22	626	5	34	358	2.721	10.615
Zuführung zur RfB	19	60	0	1	38	223	1.952
ZZR-Aufwand	24	134	1	8	104	717	2.675

Mio. Euro or percent
[1] descending order
[2] Quantile of 59 insurers

Output Data and Market

Größe	Rang[1]	HUK	Min.	5%[2]	50%[2]	95%[2]	Max.
Bestandsabbaurate	43	7,27%	4,77%	5,56%	8,10%	10,56%	12,64%
Garantie	24	6.283	141	508	4.349	39.642	183.605
Garantien und Optionen	14	14,74	0,00	0,03	4,43	71,81	102,83
HGB-DRSt ohne ZZR	24	6.755	136	546	4.344	35.412	175.798
HGB-DRSt ohne ZZR zzgl. FLV	25	6.913	398	659	5.372	40.133	181.104
HGB-Eigenkapital	10	601	12	22	149	939	1.764
Kapitalergebnis	23	367	7	22	263	1.966	9.233
latente Steuern	23	113	-16	0	67	519	2.867
Marktwert Kaptialanlagen	24	9.664	161	709	6.187	47.308	244.606
Marktwert-Bilanzsumme	24	10.197	455	867	7.302	51.228	255.484
nachhaltige Gesamtverzinsung	9	4,92%	1,85%	3,60%	4,25%	5,28%	6,85%
Nettoverzinsung	39	4,12%	3,15%	3,56%	4,34%	5,49%	8,86%
Passivduration	17	12,63	7,83	9,16	11,51	15,85	17,98
passivische Bewertungsreserven	15	1.099	-2.698	-166	216	2.587	4.569
Puffer	26	1.578	11	90	1.078	8.429	48.238
Sicherheitsmittel	23	2.516	-44	131	1.394	10.643	58.603
Sicherheitsmittelquote	17	26,66%	-2,44%	10,92%	22,52%	31,74%	37,16%
sonstige Aktiva	21	375	15	24	204	1.659	5.572
sonstige Passiva	11	1.240	23	38	446	2.527	7.970
verfügbare RfB	48	55	4	14	272	1.627	10.853
Zinsergebnis	34	12	-413	-83	15	275	2.132
zukünftige Aktionärsgewinne	23	450	-126	2	267	2.076	11.468
zukünftige pass. vt. Überschüsse	13	1.636	-152	28	429	4.057	7.155
zukünftige pass. Zinsüberschüsse	44	-537	-4.346	-1.718	-187	5	294
zukünftige Überschussbeteiligung	22	1.410	4	76	802	6.247	34.519
zukünftige Überschüsse	23	1.860	-122	99	1.070	8.323	45.986
ökonomische Eigenkapitalquote	1	9,95%	-4,41%	2,85%	5,07%	7,65%	9,95%
ökonomisches Eigenkapital	16	938	-80	34	314	2.390	10.365

Mio. Euro or percent

[1] descending order

[2] Quantile of 59 insurers

Example Summary

- Company's economic capital: 938 Mio. Euro.
- Economic capital ratio: 9,95%.
- First place of 59 German life insurers.
- Top strength: Statutory equity
- Top weakness: Hidden reserves

HUK-COBURG Leben

1.Platz

von 59
Lebensversicherern
2018

Part 3

Structural Neural Networks : Identification and Estimation

Relation to Neural Networks (1/3)

We have a structural neural network (SNN):

- Nodes correspond to variables
- Edge weights correspond to direct effects, interpretable if identified: explainable AI
- Here special case: *Linear* activation function
- “Small Data“ instead of Big Data
- Sparse structural layers
- Nonlinear optimization of quadratic target

Relation to Neural Networks (2/3)

Automatic Differentiation (AD):

- *Exact* computation of total derivatives, neither symbolic nor numeric but using computer *code*
- *No algebraic formula required*!
- Used for backpropagation in optimization of NN

Application to total derivatives E_{x}, E_{y}:

- Python modules PyTorch, autograd (DAG only)
- Input: model equations and data $\mathbf{x}$

→ autograd and our algebraic formulas identical

Relation to Neural Networks (3/3)

- In *linear* NN, all minima are global, see Kawaguchi (2016) “Deep Learning without Poor Local Minima“, MIT, i.e., identical values of target function.
- Optimization problem is ill-posed.

→ We use regularization / <u>shrinkage</u>. We have to accept estimation bias. And some coefficients are simply shrinked to their target value.

Identification (1/4)

- How to check if expert model fits to reality?
  → Estimate structural parameters from data
- One can always get from structural to reduced form parameters. But structural parameters only <u>identified</u> if reverse relation unique.
- Estimation convergence requires identification.
- Identification matrices $\mathrm{ID_x}$, $\mathrm{ID_y}$ impose zero-restrictions on $\mathrm{M_x}$, $\mathrm{M_y}$.

Identification (2/4)

- In econometrics, identification of structural linear models checked via composed matrix:

$$\mathrm{M} = \left[\mathrm{I_n} + \mathrm{M_y}, \mathrm{M_x}\right]$$

- Denote by $\mathrm{M}_{[\mathrm{j},-\mathrm{j}]}$ the matrix M, with j-th row deleted and only those columns kept, where with zero in j-th row.
- <u>Rank criterion</u>: coeffs of row j identified iff:

$$\mathrm{rk}\left(\mathrm{M}_{[\mathrm{j},-\mathrm{j}]}\right) = \mathrm{n} - 1.$$

Identification (3/4)

- Rank criterion just for observed variables.
- But in our expert model most model variables are <u>latent</u> (not observed). There is *no* general identification procedure of structural linear models for latent variables, see Bollen (1989) p. 331, or cyclic graphs, see Pearl (2017).
- Graphical Theorem 2 of Pearl (2017) not available in linear algebra form.

Identification (4/4)

- We use <u>empirical local identification</u>: Hessian (matrix of 2nd order derivatives) of target function must have full rank at the point $\mathbf{x}$ of observed data.
- Note: If a model is not identified, one could substitute model equations and check for identification of this lower dimensional model.

Estimation (1/8)

Model equations yield *theoretical* effects.
Compatible with *empirical* data?

Assumptions:

1) Linearity (constant effects, independent of $\mathbf{x}$)
2) Homogeneity (same effects for all insurers)

Con: Does *not* hold in practice, just a proxy.

Pro: Identification can easily be checked, statistical test theory well developed.

Estimation (2/8)

- Estimate structural neural network SNN
- *Identified* effects are theoretically *non-zero* partial derivatives of equation system, given by identification restrictions ID_{x}, ID_{y}
- Estimated effects averaged over all insurers.
- Modelling demeaned data:

$$\mathrm{d}\mathbf{y} = \mathbf{y} - \bar{\mathbf{y}},$$
$$\mathrm{d}\mathbf{x} = \mathbf{x} - \bar{\mathbf{x}}$$

Estimation (3/8)

- Structural form

$$\mathrm{d}\mathbf{y} = \mathrm{M_y}\mathrm{d}\mathbf{y} + \mathrm{M_x}\mathrm{d}\mathbf{x}$$

(instead of reduced form $\mathrm{d}\mathbf{y} = \mathrm{E_x}\mathrm{d}\mathbf{x}$.)

- Matrices with τ observations (insurers):

$$\mathrm{dY} = [\mathrm{d}\mathbf{y}_1, \dots, \mathrm{d}\mathbf{y}_\tau]$$
$$\mathrm{dX} = [\mathrm{d}\mathbf{x}_1, \dots, \mathrm{d}\mathbf{x}_\tau]$$

- Forecasts: $\mathrm{d}\widehat{\mathrm{Y}} = \mathrm{E_x}\mathrm{dX} = \left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1}\mathrm{M_x}\mathrm{dX}$
- Select manifest **y** via matrix S: $\mathrm{d}\widehat{\mathrm{Y}}_\mathrm{m} = \mathrm{S}\mathrm{d}\widehat{\mathrm{Y}}$

Estimation (4/8)

- Sum of whitened squared manifest errors:

$$\mathrm{SSE} = \mathrm{tr}\left(\left(\mathrm{d}\widehat{\mathrm{Y}}_{\mathrm{m}} - \mathrm{dY}_{\mathrm{m}}\right)^{\mathrm{T}} \Sigma_{\mathrm{d}\mathbf{y}_{\mathrm{m}}}^{-1} \left(\mathrm{d}\widehat{\mathrm{Y}}_{\mathrm{m}} - \mathrm{dY}_{\mathrm{m}}\right)\right)$$

- $\boldsymbol{\theta}$ is vector of non-zero parameters in $\mathrm{M_x}$, $\mathrm{M_y}$
- $\boldsymbol{\theta}_0$ theoretical values given by expert system
- Shrinking towards prior knowledge $\boldsymbol{\theta}_0$:

$$\mathrm{Shrink} = \alpha(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^{\mathrm{T}} \Sigma_{\boldsymbol{\theta}}^{-1} (\boldsymbol{\theta} - \boldsymbol{\theta}_0)$$

- <u>Target function</u>: $\mathrm{v} = \mathrm{SSE} + \mathrm{Shrink}$

Estimation (5/8)

- Covariance matrix of estimated $\widehat{\boldsymbol{\theta}}$:

$$\widehat{\Sigma}_{\boldsymbol{\theta}} = 2\widehat{\sigma}^2 \mathrm{H}_{\mathbf{v}}^{-1}$$

- H_{v} is Hessian of target function v
- Error variance: $\widehat{\sigma}^2 = SSE\left(\widehat{\boldsymbol{\theta}}\right)/(\tau - \mathrm{df}_{\mathrm{eff}})$
- *Effective* degrees of freedom (not integer due to shrinkage), with numeric derivative:

$$\mathrm{df}_{\mathrm{eff}} = \sum_{i=1}^{p} \sum_{t=1}^{\tau} \frac{\partial \widehat{\mathrm{Y}}_{\mathrm{it}}}{\partial \mathrm{Y}_{\mathrm{it}}}$$

1^{st} Derivative of Target Function

- First derivatives of SSE:

$$\frac{\partial \mathrm{SSE}}{\partial \mathrm{M_y}} = \left(\nabla \mathrm{d}\widehat{\mathrm{Y}}^{\mathrm{T}}\right) \circ \mathrm{ID_y},\ \frac{\partial \mathrm{SSE}}{\partial \mathrm{M_x}} = \left(\nabla \mathrm{d}\mathrm{X}^{\mathrm{T}}\right) \circ \mathrm{ID_x}$$

$$\nabla = 2\left(\mathrm{S}\left(\mathrm{I_n} - \mathrm{M_y}\right)^{-1}\right)^{\mathrm{T}} \Sigma_{\mathrm{d}\mathbf{y}_\mathrm{m}}^{-1}\left(\mathrm{d}\widehat{\mathrm{Y}}_\mathrm{m} - \mathrm{dY_m}\right)$$

- First derivative of Shrink:

$$\frac{\partial \mathrm{Shrink}}{\partial \boldsymbol{\theta}} = 2\alpha\Sigma_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\theta} - \boldsymbol{\theta}_0)$$

$\rightarrow \frac{\partial \mathrm{v}}{\partial \boldsymbol{\theta}}$: elementwise combine $\frac{\partial \mathrm{SSE}}{\partial \mathrm{M_y}}$ and $\frac{\partial \mathrm{Shrink}}{\partial \boldsymbol{\theta}}$

2^{nd} Derivative of Target Function

- Second derivative of SSE: elementwise derivation of algebraic Hessian, using $\mathrm{ID_x}$, $\mathrm{ID_y}$, see Bartel (2019), sec. 5.2.:

$$\mathrm{H_{SSE}} = \frac{\partial \mathrm{SSE}^2}{\partial \boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}} = \cdots$$

- Second derivative of Shrink:

$$\mathrm{H_{Shrink}} = \frac{\partial \mathrm{Shrink}^2}{\partial \boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}} = 2\alpha\Sigma_{\boldsymbol{\theta}}^{-1}$$

$\rightarrow \mathrm{H_v} = \mathrm{H_{SSE}} + \mathrm{H_{Shrink}}$

Estimation (6/8)

Exact algebraic Hessian $\mathrm{H_v}$ of target v:

1) Local identification if $\mathrm{H_v}$ has full rank at $\mathbf{x}$
2) Yields covariance matrix of estimates $\widehat{\boldsymbol{\theta}}$
3) Improve input for nonlinear optimization

Estimation (7/8)

- *Iterative* Newton-Raphson optimization:

$$\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \mathrm{r}\mathrm{H}_{\mathrm{v}}^{-1}\frac{\partial \mathrm{v}}{\partial \boldsymbol{\theta}}$$

- Theoretical starting values $\boldsymbol{\theta}_0$
- Optimization in Python (SciPy optimize), passing algebraic gradient $\frac{\partial \mathrm{v}}{\partial \boldsymbol{\theta}}$ and Hessian H_{v}, both satisfying restrictions ID_{x}, ID_{y}

Estimation (8/8)

- Generalized cross validation criterion for optimal shrinkage parameter α:

$$\mathrm{GCV} = \frac{\mathrm{SSE}}{\tau}\left(\frac{\tau}{\tau - \mathrm{df}_{\mathrm{eff}}}\right)^2$$

- Higher α reduces $\mathrm{df}_{\mathrm{eff}}$ and GCV but increases SSE and GCV. Find optimal α, minimizing GCV using grid search.

Model Calibration (1/2)

- Given the estimated direct effects $\widehat{\boldsymbol{\theta}}$ we can calibrate our theoretical expert model.
- Adapt theoretical model if significant deviations $\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0$ given covariance matrix $\hat{\Sigma}_{\boldsymbol{\theta}}$
- Calibrate identified structural direct effects $\mathrm{M_x}$, $\mathrm{M_y}$ not total effects $\mathrm{E_x}$, $\mathrm{E_y}$
- Estimation just gives information on direct effects not restricted to zero by identification

Model Calibration (2/2)

Calibration of insurance ratings model:

- We added a spread of 1% to risk-free rate R when computing the market value of technical provisions (MWDR) in model equation 14.
- Accounts for Solvency II yield curve extrapolation towards ultimate forward rate (UFR)
- Increasing fit to published solvency ratios (without volatility adjustment, transitional measures).

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